Projective generation and smoothness of congruences of order 1*

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Abstract

In this paper we give the projective generation of congruences of order 1 of r-dimensional projective spaces in $\mathbb{P}^N$ from their focal loci. In a natural way, this construction shows that the corresponding surfaces in the grassmannian are the Veronese surface, and rational ruled surfaces eventually with singularities. We characterize when these surfaces are smooth, recovering and generalizing a Ziv Ran's result.

Introduction

Let $\mathbb{P}^N$ be the N–dimensional complex projective space and $G(r,N)$ be the Grassmannian of $\mathbb{P}^r$'s in $\mathbb{P}^N$. The study of surfaces $\Sigma \subset G(r,N)$ –called *congruences*– is a classical topic. The algebraic equivalence class of Σ consists of a pair of integers named *order* and *class*: the order of the congruence is the number of elements of Σ meeting a general $(N-r-2)$–dimensional space. In 1866, Kummer [8] achieved the classification of congruences of lines in $\mathbb{P}^3$ of order 1. A classification of congruences of order 1 when $r \geq 1$ and $N \geq 3$ was obtained independently by Ziv Ran in [10] using Bertini's theorem. Kummer's basic idea was studying the set of *Fundamental Points* of the congruence; that is, the points lying on an infinite number of its lines. Recently F. Zak, A. Inshakov, S. Lvovski and A. Obolmkov in [13] have shown that the classification of congruences of order 1 in $G(1,3)$ can be rigorously done with elementary arguments.

Since every fundamental point of a congruence is a focal point, one may ask whether the focal methods provide sufficient information to classify congruences. The focal method was introduced by Corrado Segre [11], and it was used successfully in [9] to obtain a birrational classification of congruences of planes; the modern reference here is [1]. In this article we expose a classification of congruences of order 1 for all $r \geq 1$ and $N \geq 3$ by using the focal method. The advantage of using it is that it provides locally the construction of Σ from its focal locus $F(\Sigma)$. In particular, the method allows us to decide the smoothness of the surface parametrizing it in the grassmannian recovering and generalizing the Ziv Ran's result about smooth surfaces of order 1 in $G(1,3)$.

The paper is organized as follows: In sections 1 and 2 we summarize some properties about surfaces in grassmannians and their focal loci. For a congruence of order 1 we show that every focal

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point is a fundamental point. Another important ingredient here is the relation between sections of a congruence in $\mathbb{P}^N$ with the projections of the corresponding surface in the grassmannian, reducing some proofs to the case $r = 1$. Excepting the family of $\mathbb{P}^r$'s containing a fixed $\mathbb{P}^{r-1}$ in $\mathbb{P}^{r+2}$, the focal locus $F(\Sigma)$ has dimension r, obtaining three possibilities: either it is irreducible of degree 1 (case III) or 3 (case I), or $F(\Sigma) = \pi \cup X$, being π a projective space and X a rational scroll (case II). It is also shown that if $F(\Sigma)$ contains a projective space, Σ is a rational ruled surface. Section 3 deals with the case I: such a congruence is parametrized by a Veronese surface; we also prove that there are only three fixed point free models, and the remaining are constructed as cones over one of those. In section 4, we show the projective generation of a congruence Σ in the case II from $F(\Sigma) = \pi \cup X$, studying the normal models of the scroll X; we characterize the singular locus of Σ, and prove the existence of such surfaces for each invariant e. In section 5 it is proved that a congruence in the case III is parametrized by a rational cone, as well as its existence. We also give a characterization of such congruences through a morphism between projective spaces. Finally, in section 6, we resume all the results about smoothness of surfaces of order 1 in grassmannians.

1 About Congruences of Order 1

Let $\mathbb{P}^N$ be the N–dimensional projective complex space and $G(r, N)$ be the Grassmannian of r–dimensional subspaces in $\mathbb{P}^N$. We begin compiling some basic facts about $G(r, N)$ (see [7] for more details):

It is well known that the cohomology group $H^i(G(r, N), \mathbb{Z})$ is 0 if $i \notin [0, 2(N - r)(r + 1)]$ and the direct sum

$$H^*(G(r, N), \mathbb{Z}) = \bigoplus_i H^i(G(r, N), \mathbb{Z})$$

is a graded ring with the cup–product $\cup$. We can assign the cohomology class $[X]$ to each closed subset $X \subset G(r, N)$, being satisfied the next properties:

- Two algebraically equivalent subvarieties have the same cohomology class.
- If $X \subset G(r, N)$ is irreducible $((r + 1)(N - r) - p)$–dimensional, then $[X] \in H^{2p}(G(r, N), \mathbb{Z})$.
- For each two subvarieties $X, Y \subset G(r, N)$, $[X \cap Y] = [X] \cup [Y]$.

Let $A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_r$ be a strictly increasing sequence of subspaces of $\mathbb{P}^N$, $a_i := \dim A_i$, $i = 0, \ldots, r$, and $\Omega(A_0, \ldots, A_r)$ denote the next subvariety of the grassmannian:

$$\Omega(A_0, \ldots, A_r) = \{\sigma \in G(r, N) \ / \dim(\mathbb{P}^r(\sigma) \cap A_i) \geq i,\ i = 0, \ldots, r\}$$

Such a subvariety is called a *Schubert cycle* in $G(r, N)$. Regarding $G(r, N)$ as a projective variety through the Plücker's embedding, Schubert cycles can be obtained cutting it with some linear subspaces; for instance, the Schubert cycle $\Omega_r \subset G(1, 3)$ parametrizing the lines in $\mathbb{P}^3$ meeting a given one r consists of the intersection $\Omega_r = G(1, 3) \cap T_{G(1,3),[r]}$.

Given two Schubert cycles $\Omega(A_0, \ldots, A_r)$, $\Omega(B_0, \ldots, B_r)$, they are algebraically equivalent when $\dim A_i = \dim B - i$ for every i; thus the cohomology class of $\Omega(A_0, \ldots, A_r)$ depends only of

the integers $a_i := \dim A_i$, $i = 0, \ldots, r$, and we denote it by $\Omega(a_0, \ldots, a_r)$. They play an important role in the cohomology of $G(r, N)$, in fact:

Theorem 1.1 *$H^*(G(r,N),\mathbb{Z})$ is a free abelian group with basis*

$$\{\Omega(a_0, \ldots, a_r) \,/\, 0 \leq a_0 < a_1 < \ldots < a_r \leq N\}.$$

$\blacksquare$

Moreover, each $\Omega(A_0, \ldots, A_r)$ is irreducible of dimension $\sum_{i=0}^{r}(a_i - i)$, so we have the next corollary:

Corollary 1.2 *For every integer p, $H^{2p}(G(r,N),\mathbb{Z})$ is a free abelian group generated by the elements $\Omega(a_0, \ldots, a_r)$ with $p = [(r+1)(N-r) - \sum_{i=0}^{r}(a_i - i)]$. Each $H^{2p+1}(G(r,N),\mathbb{Z})$ is 0.* $\blacksquare$

Example 1.3 By the above, we can calculate the basis of the cohomology groups of codimension 2 and dimension 2:

$$\begin{aligned}
&H^{2(N-r)(r+1)-4}(G(r,N),\mathbb{Z}) = \langle \Omega_1 :\equiv \Omega(0, 1, \ldots r-1, r+2),\\
&\Omega_2 :\equiv \Omega(0, 1, \ldots r-2, r, r+1)\rangle\\
&H^4(G(r,N),\mathbb{Z}) = \langle \Omega^1 :\equiv \Omega(N-r-2, N-r+1, N-r+2, \ldots N),\\
&\Omega^2 :\equiv \Omega(N-r-1, N-r, N-r+2 \ldots N)\rangle
\end{aligned}$$

Denoting by $L^i \subset \mathbb{P}^n$ a fixed generic i–dimensional subspace, the elements above correspond to the classes of the Schubert cycles:

$$\begin{aligned}
\Omega_1 &= [\{\sigma \in G(r,N) \,/ L^{r-1} \subset \mathbb{P}(\sigma) \subset L^{r+2}\}]\\
\Omega_2 &= [\{\sigma \in G(r,N) \,/ L^{r-2} \subset \mathbb{P}(\sigma) \subset L^{r+1}\}]\\
\Omega^1 &= [\{\sigma \in G(r,N) \,/ \mathbb{P}(\sigma) \cap L^{N-r-2} \neq \emptyset\}]\\
\Omega^2 &= [\{\sigma \in G(r,N) \,/ \dim(\mathbb{P}(\sigma) \cap L^{N-r}) \geq 1\}]
\end{aligned}$$

and we can compute the products $\Omega_i \cup \Omega^j$ as the classes of the intersections of such Schubert cycles; thus $\Omega_i \cup \Omega^j = 0$ if $i \neq j$ and $\Omega_i \cup \Omega^j = 1$ if $i = j$. Hence the cohomology class of a 2–dimensional irreducible subvariety $\Sigma \subset G(r,N)$ is determined by the number of points of intersection of Σ with a generic Schubert cycle in the class of Ω^1 (the number of elements of Σ cutting a generic $(N-r-2)$–dimensional space), and with another one in the class of Ω_2 (the number of elements of Σ meeting a generic $(N-r)$–dimensional space in lines); these numbers are called, respectively, *order* ($\mathrm{ord}(\Sigma)$) and *class* ($\mathrm{cl}(\Sigma)$) of Σ. The pair $(\mathrm{ord}(\Sigma, \mathrm{cl}(\Sigma))$ is called *bidegree* of Σ. Moreover, regarding Σ as a projective variety through the Plücker embedding of the grassmannian, the degree of Σ is exactly $\mathrm{ord}(\Sigma) + \mathrm{cl}(\Sigma)$.

We wish to investigate 2–dimensional families of r–dimensional subspaces in $\mathbb{P}^N$, called classically *congruences*. From now on, Σ will denote a 2–dimensional reduced and irreducible closed subvariety of $G(r,N)$ and $V(\Sigma)$ its projective realization,

$$V(\Sigma) := \bigcup_{\sigma \in \Sigma} \mathbb{P}^r(\sigma) \tag{1}$$

The objective of this work is the classification of congruences of order 1; by the next lemma we will be reduced to studying the case $N = r + 2$.

Lemma 1.4 *Let $\Sigma \subset G(r,N)$ be a congruence. If $\operatorname{ord}(\Sigma) = 1$, then there exist a $(r+2)$–dimensional projective subspace $A \subset \mathbb{P}^N$ such that $\mathbb{P}^r(\sigma) \subset A$ for all $\sigma \in \Sigma$.*

Proof: Since Σ is irreducible, $V(\Sigma)$ is irreducible too. Let $B \subset \mathbb{P}^N$ be a generic $(N-r-2)$–dimensional subspace. As $\operatorname{ord}(\Sigma) = 1$ we have $B \cap V(\Sigma) = B \cap \mathbb{P}^r(\sigma) \neq \emptyset$ for only one $\sigma \in \Sigma$. It follows that $\dim V(\Sigma) = r + 2$ and $\deg V(\Sigma) = 1$. We finish taking $A := V(\Sigma) \simeq \mathbb{P}^{r+2}$. ■

According to this lemma, we will only consider $\Sigma \subset G(r, r+2)$, using the next notation for the incidence variety:

$$\begin{array}{ccc} \mathbf{P}^{r+2} \times \Sigma \supset \mathcal{I}_\Sigma := \{(P,\sigma) \in \mathbf{P}^{r+2} \times \Sigma;\ P \in \mathbf{P}^r(\sigma)\} & \xrightarrow{p} & \mathbf{P}^{r+2} \\ \downarrow q & & \\ \Sigma & & \end{array}$$

Definition 1.5 Let $\Sigma \subset G(r, r+2)$ be a congruence. A point $P \in \mathbb{P}^{r+2}$ is called *fundamental* for Σ if $\dim(p^{-1}(P)) \geq 1$. The congruence Σ is called *degenerate* iff $\dim V(\Sigma) < r+2$, that is iff every point in $V(\Sigma)$ is fundamental. For instance, a congruence of order 1 is nondegenerate.

Given a congruence Σ in $\mathbb{P}^{r+2}$ and a projective subspace $L \subset \mathbb{P}^{r+2}$, we can obtain a congruence in L cutting it with each element of Σ. The next results provide some properties of such sections.

Lemma 1.6 *Let $\Sigma \subset G(r, r+2)$ be a congruence of order $\operatorname{ord}(\Sigma)$ and $L \subset \mathbb{P}^{r+2}$ be a $(l+2)$–dimensional generic subspace, $l \geq 1$; then Σ does not meet $\Omega_L := \{\sigma \in G(r, r+2) \,/\, \dim L \cap \mathbb{P}^r(\sigma) \geq l+1\}$, which is a codimension $l+2$ Schubert cycle; thus the projection*

$$\rho_L : \Sigma \subset G(r, r+2) \longrightarrow G(l, L), \quad \pi_L(\sigma) := [\mathbb{P}^r(\sigma) \cap L]$$

is regular and has as image a surface Σ_L such that $\operatorname{ord}(\Sigma_L) = \operatorname{ord}(\Sigma)$, $\operatorname{cl}(\Sigma_L) = \operatorname{cl}(\Sigma)$.

Proof: Consider the incidence variety

$$\mathcal{I} := \{(L,\sigma) \,/\, \dim(L \cap \mathbb{P}^r(\sigma)) \geq l+1\} \subset G(l+2, r+2) \times \Sigma$$

with projections p_1 and q_1. Given $\sigma \in \Sigma$, $q_1^{-1}(\sigma) \equiv \{L \in G(l+2, r+2) \,/\, \dim(\mathbb{P}^r(\sigma) \cap L) \geq l+1\}$ is a codimension $l+2$ Schubert cycle (because its cohomology class is $\Omega(r-1-l, \ldots, r-1, r, r+2)$), so $\dim \mathcal{I} = \dim \Sigma + \dim q_1^{-1}(\sigma) = 2 + \dim G(l+2, \mathbb{P}^{r+2}) - l - 2 = \dim G(l+2, \mathbb{P}^{r+2}) - 1$. Therefore $p_1(\mathcal{I}) \subset G(l+2, r+2)$ is a proper closed subset, and if $L \in G(l+2, r+2) \setminus p_1(\mathcal{I})$, then $\dim(L \cap \mathbb{P}^r(\sigma)) = l$ for all $\sigma \in \Sigma$. It follows that the map ρ_L defined above is regular. Furthermore it is birrational: a generic point $P \in \mathbb{P}^{r+2}$ is contained in a finite number of elements of the congruence, $\mathbb{P}^r(\sigma_1), \ldots, \mathbb{P}^r(\sigma_n)$; a generic l–dimensional subspace through P, $W \subset \mathbb{P}^r(\sigma_1)$ will not be contained in another $\mathbb{P}^r(\sigma_i)$; so taking $L \cap \mathbb{P}^r(\sigma_1) =: W$, we will have $L \cap \mathbb{P}^r(\sigma_1) = W \neq L \cap \mathbb{P}^r(\sigma_2), \ldots, L \cap \mathbb{P}^r(\sigma_n)$; thus $\rho_L^{-1}(W) = \{\sigma_1\}$, so $\deg \rho_L = 1$.

By construction, the equality $\operatorname{ord}(\Sigma_L) = \operatorname{ord}(\Sigma)$ holds; by the birrationality of ρ_L $\deg(\Sigma_L) = \deg(\Sigma)$ and so $\operatorname{cl}(\Sigma_L) = \operatorname{cl}(\Sigma)$ holds too. ■

Remark 1.7 Furthermore, regarding $G(r, r+2)$ and $G(l, l+2)$ as projective varieties through the corresponding Plücker embeddings, the map ρ_L is easily shown to be the projection from the space generated by the Schubert cycle $\{\sigma \in G(r, r+2) \,/\, \dim(\mathbb{P}^r(\sigma) \cap L) \geq 2\}$.

Lemma 1.8 *Let* $\Sigma \subset G(r, r+2)$ *be a nondegenerate congruence, and* $F(\Sigma) \subset \mathbb{P}^{r+2}$ *its fundamental locus. Then* $\dim F(\Sigma) \leq r$.

Proof: Suppose the lemma were false. We could find an $(r+1)$–dimensional irreducible component $C \subset F(\Sigma)$. Considering the restriction $p^{-1}(C) \subset \mathcal{I}_\Sigma \longrightarrow C \subset \mathbb{P}^{r+2}$, we would have $\dim p^{-1}(C) = r+2$, and so $p^{-1}(C) = \mathcal{I}_\Sigma$. Consequently Σ would be degenerate. ∎

Lemma 1.9 *Under the hypotheses of (1.6) and with the above notation, it is verified the equality* $F(\Sigma) \cap L = F(\Sigma_L)$.

Proof: By construction $F(\Sigma) \cap L \supset F(\Sigma_L)$. If $P \in F(\Sigma) \cap L \setminus F(\Sigma_L)$, P would be contained in an irreducible family of elements of Σ with the same trace on L. An l–dimensional subspace contained in infinite elements of Σ is called *fundamental*. Suppose Σ is nondegenerate (the degenerate case is trivial). It is sufficient to show that, if D is an irreducible component of $F(\Sigma)$ and if L is generic, $D \cap L$ does not contain fundamental subspaces. On the contrary, suppose $D \cap L$ contains fundamental subspaces for all L; by (1.8) $\dim D \leq r$, so D would be an r–dimensional projective space whose l–dimensional subspaces are all fundamental. Considering the incidence variety:

$$\mathcal{J} := \{(l, \sigma) \,/\, l \subset \mathbb{P}^r(\sigma) \cap D\} \subset G(l, D) \times \Sigma$$

with projections p_2 and q_2, we have $\dim \mathcal{J} = 2 + \dim q_2^{-1}(\sigma) \leq 2 + (l+1)(r-1-l)$, because $q_2^{-1}(\sigma) \cong G(1, \mathbb{P}^r(\sigma) \cap D)$. Thus $\dim \mathcal{J} \leq 2 + (l+1)(r-1-l) \leq (l+1)(r-l) = \dim G(l, D)$, so either p_2 is not surjective, or it has generically finite fibers. In both cases, the generic l–dimensional subspace is not fundamental, a contradiction. ∎

Definition 1.10 A point $P \in \mathbb{P}^r(\sigma)$ is called *fixed point* of the congruence Σ when $P \in \mathbb{P}^r(\sigma)$ for all $\sigma \in \Sigma$.

Remark 1.11 The locus of fixed points of a congruence Σ is a linear subspace that we will denote by $T(\Sigma) \subset \mathbb{P}^{r+2}$, and Σ lies on the Schubert's cycle:

$$\Omega_{T(\Sigma)} := \{\sigma \in G(r, r+2)/\mathbb{P}^r(\sigma) \supset T(\Sigma)\}$$

Being $L \subset \mathbb{P}^{r+2}$ a generic complementary subspace of $T(\Sigma)$, $k+2 := \dim L = r+2-\dim T(\Sigma)-1$, the maps

$$\begin{aligned} \rho_L &: \sigma \in \Omega_{T(\Sigma)} \longmapsto [\mathbb{P}^r(\sigma) \cap L] \in G(k, k+2) \\ \rho^{T(\Sigma)} &: \tau \in G(k, k+2) \longmapsto [\mathbb{P}^k(\tau) + T(\Sigma)] \in \Omega_{T(\Sigma)} \end{aligned} \tag{2}$$

provide an isomorphism $G(k, k+2) \cong \Omega_{T(\Sigma)} \subset G(r, r+2)$. Clearly, the congruence $\rho_L(\Sigma) \cong \Sigma$ has no fixed points; roughly speaking, $\rho_L(\Sigma)$ is the same surface as Σ living in a grassmannian of lesser dimension. Therefore *it suffices to study congruences without fixed points.*

2 Focal Locus of a Congruence in $G(r, r+2)$

Definition 2.1 Let $\Sigma \subset G(r, r+2)$ be a congruence. Being $\sigma \in \Sigma$ smooth, a point $P \in \mathbb{P}^r(\sigma)$ is called *focal* if $(dp)_{(P,\sigma)}$ is not injective. If $\ker((dp)_{(P,\sigma)}) \supset \langle v \rangle$ for some $v \in q^*T_{\Sigma,\sigma}$, $v \neq 0$, we say that P is focal for the direction $\langle v \rangle$.

Proposition 2.2 *Let $\Sigma \subset G(r, r+2)$ be a congruence. If $P \in \mathbb{P}^{r+2}$ is a fundamental point of Σ, then P is focal for every $\sigma \in \Sigma$ smooth with $P \in \mathbb{P}^r(\sigma)$. Furthermore, if $P \in \mathbb{P}^r(\sigma)$ is focal and if $\operatorname{ord}(\Sigma) = 1$, then P is fundamental.*

Proof: Being P fundamental, there is an irreducible curve $C \in \Sigma$ such that $P \in \sigma$ for all $\sigma \in C$. Taking a smooth point $\sigma \in C$ and regarding the curve $\{P\} \times C \subset \mathcal{I}_\Sigma$, we have $p(\{P\} \times C) = P$. Assuming that σ is smooth in Σ we can write $(dp)_{(P,\sigma)}(T_{\{P\}\times C,(P,s)}) = 0$. Therefore $P \in \mathbb{P}^r(\sigma)$ is focal.

For the second part we use that $\operatorname{ord}(\Sigma) = \deg(p)$. If $\operatorname{ord}(\Sigma) = 1$, the map p is dominant and $p(\mathcal{I}_\Sigma) = \mathbb{P}^{r+2}$ is normal. Hence $\deg(p) = \sum_{p(x)=P} m_x(p)$ for every nonfundamental point P, being $m_x(p) := \dim_{\mathbb{C}}(\mathcal{O}_{\mathcal{I}_\Sigma,x}/p^*\mathfrak{m}_P)$ for $\mathfrak{m}_P$ the maximal ideal of the local ring $\mathcal{O}_{\mathbb{P}^{r+2},P}$. Thus, if $\operatorname{ord}(\Sigma) = 1$, $m_x(p) > 1$ forces P to be fundamental. The proof is completed with the next lemma. ■

Lemma 2.3 *Let $\Sigma \subset G(r, r+2)$ be a congruence and denote $x := (P, \sigma) \in \mathcal{I}_\Sigma$. x is focal if and only if $m_x(p) \geq 2$.*

Proof: Regarding the diagram

$$\begin{tikzcd} p^*\mathfrak{m}_P \arrow[r] \arrow[d] & \mathcal{O}_{\mathcal{I}_\Sigma,x} \arrow[r] \arrow[d, equal] & \mathcal{O}_{\mathcal{I}_\Sigma,x}/p^*\mathfrak{m}_P \arrow[d] \\ \mathfrak{m}_x \arrow[r] & \mathcal{O}_{\mathcal{I}_\Sigma,x} \arrow[r] & \mathbb{C} \end{tikzcd}$$

it is easily seen that $\dim_{\mathbb{C}}(\mathcal{O}_{\mathcal{I}_\Sigma,x}/p^*\mathfrak{m}_P) = \dim_{\mathbb{C}}(\mathfrak{m}_x/p^*\mathfrak{m}_P) + 1$. Consider the next diagram:

$$\begin{tikzcd} \mathfrak{m}_P^2 \arrow[r] \arrow[d, "p^*"] & \mathfrak{m}_P \arrow[r] \arrow[d, "p^*"] & \mathfrak{m}_P/\mathfrak{m}_P^2 = (T_{\mathbb{P}^{r+2},P})^* \arrow[d, "(dp)_x^*"] \\ \mathfrak{m}_x^2 \arrow[r] & \mathfrak{m}_x \arrow[r] & \mathfrak{m}_x/\mathfrak{m}_x^2 = (T_{\mathcal{I}_\Sigma,x})^* \end{tikzcd}$$

By *Nakayama's lemma* p^* is surjective iff $(dp)_x^*$ is surjective, that is iff $(dp)_x$ is injective. Hence, x is focal iff $\dim_{\mathbb{C}}(\mathfrak{m}_x/p^*\mathfrak{m}_P) > 0 \iff m_x(p) > 1$. ■

We continue with some basic facts about the focal locus of a congruence in $G(r, r+2)$; we refer the reader to [9] for more details in the case $r = 2$, but the proofs also work for all r. Let $\Sigma \subset G(r, r+2)$ be a congruence and $\sigma \in \Sigma$ an smooth point. For each $P \in \mathbb{P}^r(\sigma)$ there is an isomorphism $T_{\mathcal{I}_\Sigma,(P,\sigma)} \cong T_{\Sigma,\sigma} \oplus T_{\mathbb{P}^r(\sigma),P}$.

The *focal locus at* $\mathbb{P}^r(\sigma)$ can be calculated through the *Characteristic Map of Kodaira–Segre–Spencer*:

$$\chi : T_{\Sigma,\sigma} \longrightarrow H^0(\mathcal{N}_{\mathbb{P}^r(\sigma),\mathbb{P}^{r+2}}) \cong H^0(\mathcal{O}_{\mathbb{P}^r(\sigma)}(1)) \oplus H^0(\mathcal{O}_{\mathbb{P}^r(\sigma)}(1)) \tag{3}$$

being $\chi(v) = 0$ the equations of the focal locus of Σ at $\mathbb{P}^r(\sigma)$ in the direction $\langle v\rangle \subset T_{\Sigma,\sigma}$. An easy computation shows that:

- The focal locus at $\mathbb{P}^r(\sigma)$ in a direction $\langle v\rangle$ is a projective subspace of dimension $r-2$ or $r-1$ and we denote it by $Z(\chi(v))$. Taking a base $\{v_1, v_2\}$ in $T_{\Sigma,\sigma}$, whose images through χ are (f_{11}, f_{12}) and (f_{21}, f_{22}) respectively, $Z(\chi(\lambda v_1 + \mu v_2))$ will be the set of points $P \in \mathbb{P}^r(\sigma)$ satisfying the equations:
$$\left.\begin{array}{l} \lambda f_{11}(P) + \mu f_{21}(P) = 0 \\ \lambda f_{12}(P) + \mu f_{22}(P) = 0 \end{array}\right\}$$

- The focal locus at $\mathbb{P}^r(\sigma)$ is a quadric $Q(\sigma) = \bigcup_{v\in T_{\Sigma,\sigma}} Z(\chi(\sigma))$, and it is given by the equation
$$\det \begin{pmatrix} f_{11}(P) & f_{21}(P) \\ f_{12}(P) & f_{22}(P) \end{pmatrix} = 0 \tag{4}$$
It is reducible if and only if there exist a direction $\langle v\rangle \subset T_{\Sigma,\sigma}$ such that $\dim Z(\chi(v)) = r-1$. Such directions are called *developable*, and the number of them is fixed in an open subset of Σ; this number can be 0, 1, 2 (distinct or coincident) or ∞ (if all direction is developable).

- If $Q(\sigma)$ is irreducible, it contains a 1–dimensional family of $(r-2)$–dimensional projective subspaces. Therefore $\operatorname{rank} Q(\sigma) \leq 4$ (see for instance [3], book V, page 100). $Q(\sigma)$ will be a cone with vertex a subspace $C(\sigma) \subset \mathbb{P}^r(\sigma)$ over an smooth conic or an smooth quadric surface. In both cases the focal loci in each direction are subspaces of maximal dimension contained in $Q(\sigma)$, and so the intersection of two of them is $C(\sigma)$. Hence $C(\sigma)$ is the set of points focal for all direction.

- Let $\Delta \subset G(r, r+2)$ be a reduced and irreducible curve. We say that Δ is a *developable system* if the focal locus in each $\mathbb{P}^r(\sigma)$, $\sigma \in \Delta$ has dimension $r-1$. Following [9], Δ parametrizes a family of the next kind: Cone with vertex a subspace $\mathbb{P}^k$ over the family of $(r-k-1)$–osculating spaces to an irreducible curve $\mathcal{C} \subset \mathbb{P}^{r-k+1}$, $k = -1, 0, \ldots, r-1$. For instance, a 1–dimensional family of $\mathbb{P}^r$'s lying in an $(r+1)$–dimensional space is always developable.

- A nondegenerate congruence $\Sigma \subset G(r, r+2)$ has reducible generic focal quadric iff by the generic $\sigma \in \Sigma$:
  - is passing 1 developable system, or
  - are passing 2 distinct developable systems, or
  - are passing 2 coincident developable systems, or
  - are passing infinite developable systems.

Being $\Sigma \subset G(r, r+2)$, denote $F_1(\Sigma) \subset \mathbb{P}^{r+2}$ its focal variety, that is, the projective realization of the family of focal quadrics, which is defined in an open set $U \subset \Sigma$, $\mathcal{F}_1(U) \longrightarrow U$. In general, the fundamental locus $F(\Sigma)$ is a subvariety of $F_1(\Sigma)$ (see [2]). In the case $\mathrm{ord}(\Sigma) = 1$, (2.2) clearly forces the equality $F_1(\Sigma) = F(\Sigma)$. Moreover, by (1.8) $\dim F(\Sigma) \leq r$, $\dim F(\Sigma) < \dim F_1(U) = r+1$. The generic focal quadric can be irreducible or not, so either $F(\Sigma)$ is irreducible, or it is the projective realization of two 2–dimensional families of $\mathbb{P}^{r-1}$'s, having at least 2 irreducible components.

Example 2.4 Let C_1, C_2 be two irreducible curves in $\mathbb{P}^3$. We write $\Sigma(C_1, C_2)$ for the set of lines joining points of both curves, and $\Sigma(C_1)$ the family of secant lines to C_1. Clearly the focal locus of these congruences contains the base curves (furthermore, they will be exactly the focal locus in some cases). If C_1 and C_2 are not coplanar, then $\Sigma(C_1, C_2)$ is nondegenerate, and if C_1 is not a plane curve, then $\Sigma(C_1)$ is nondegenerate. We will show later when these congruences have order 1.

In the next two results we examine the focal locus of a congruence of order 1, concretely its irreducible components and degree.

Proposition 2.5 *If the focal locus of a congruence Σ of order 1 is reducible, $F(\Sigma) = C_1 \cup C_2$, then both components have dimension r and one of them is linear.*

Proof: We have shown that the number of irreducible components of $F(\Sigma)$ is at most two, with dimensions lesser than or equal to r by (1.8). Let C_1, C_2 be the irreducible components of $F(\Sigma)$. It is sufficient to show that if $F(\Sigma)$ is reducible, its section with the generic 3–dimensional space consists of a line and an irreducible curve. Thus using (1.6) and (1.9), we can suppose $r = 1$. If $\dim C_i = 0$ for some i, then Σ would be the family of lines passing by C_i, whose focal locus is the point C_i, so C_1 and C_2 are irreducible curves. Every line of Σ contains a point in each curve, so $\Sigma = \Sigma(C_1, C_2)$, and $\Sigma \cap \Sigma(C_1)$ and $\Sigma \cap \Sigma(C_2)$ are closed subsets in Σ (if $\Sigma = \Sigma(C_1)$, then every line of Σ would contain at least three focal points, two of C_1 and one of C_2, so the focal quadric at each line must be the whole line; equivalently, the congruence would be degenerate, so $\mathrm{ord}(\Sigma) = 0$, false.). Being $K \subset \mathbb{P}^3$ a generic plane, it contains $\mathrm{cl}(\Sigma)$ generators of Σ, none belonging to $\Sigma(C_1)$ nor $\Sigma(C_2)$. Hence every generator in K meets C_1 and C_2 exactly one time. If $\deg C_1, \deg C_2 \geq 2$, we could take $l = P_1P_2$, $l' = Q_1Q_2$, $P_i, Q_i \in C_i$, $P_i \neq Q_i$. The point $P = l \cap l'$ could not lie on C_1, C_2, but as $\mathrm{ord}(\Sigma) = 1$, P should be fundamental, a contradiction. ■

Proposition 2.6 *If the focal locus $F(\Sigma)$ of a congruence of order 1 is irreducible, one of these possibilities is true:*

1. *$F(\Sigma)$ is an $(r-1)$–dimensional projective space and Σ is the family of $\mathbb{P}^r$'s containing $F(\Sigma)$.*
2. *$\dim F(\Sigma) = r$, and then:*
   - *(a) either $F(\Sigma)$ is a projective space,*
   - *(b) or $F(\Sigma)$ is an irreducible variety of degree 3.*

Proof: By hypothesis $F(\Sigma)$ is irreducible, containing the focal quadrics. If $\dim F(\Sigma) \leq r-1$, then $F(\Sigma) = Q(\sigma) \subset \mathbb{P}^r(\sigma)$ for all $\sigma \in \Sigma$. Therefore $F(\Sigma)$ is an $(r-1)$–dimensional projective space.

Suppose $\dim F(\Sigma) = r$. Using (1.6) and (1.9) we are reduced to the case $r = 1$. If $\deg F(\Sigma) = n > 1$, then $\Sigma = \Sigma(F(\Sigma))$: let $\sigma \in \Sigma$ be generic and $P \in F(\Sigma) \cap \mathbb{P}^1(\sigma) \neq \emptyset$; since $n > 1$, a generic plane $\pi \supset \mathbb{P}^1(\sigma)$ contains another point $Q \in F(\Sigma)$, $Q \neq P$. A 1–dimensional family of lines of Σ is passing by Q, forming a cone of degree ≥ 1; one of its generators $\mathbb{P}^1(\tau)$ lies on π, cutting $\mathbb{P}^1(\sigma)$ in a fundamental point $R \in F(\Sigma)$, and $\mathbb{P}^1(\sigma) = PR$.

Since $\Sigma = \Sigma(F(\Sigma))$, necessarily $\deg F(\Sigma) \geq 3$ (otherwise, $F(\Sigma)$ would be planar, and Σ would be degenerate). Suppose, contrary to our claim, that $\deg F(\Sigma) \geq 4$. The generic plane $\pi \subset \mathbb{P}^3$ cuts $F(\Sigma)$ in $n \geq 4$ points such that any 3 of them are not collinear (by the Trisecant Lemma). Taking four of them, the diagonal points of the square they form must be fundamental; that is, they belong to $F(\Sigma)$; hence there are 3 collinear points of $F(\Sigma) \cap \pi$, a contradiction. ■

We study now the case where $F(\Sigma)$ (irreducible or not) contains a projective space.

Lemma 2.7 *Let π be an r–dimensional projective space and consider the Schubert's cycle:*

(5) $$\Omega_\pi := \{\sigma \in G(r, r+2) \,/\, \dim \mathbb{P}^r(\sigma) \cap \pi \geq r-1\} \subset G(r, r+2)$$

Ω_π is a cone with vertex $[\pi]$ over a Segre Variety of the form $\mathbb{P}^r \times \mathbb{P}^1 \subset \mathbb{P}^{2r+1}$

Proof: Regard $H_\pi = \{[H] \in \mathbb{P}^{r+2^*} \,/\, H \supset \pi\} \cong \mathbb{P}^1$ and the projection

(6) $$\phi : \Omega_\pi \setminus \{[\pi]\} \longrightarrow \pi^* \times H_\pi, \quad \phi(\sigma) := (\mathbb{P}^r(\sigma) \cap \pi, \mathbb{P}^r(\sigma) + \pi)$$

whose fibers are $\phi^{-1}(h, H) = \Omega_{h,H} = \{\sigma \in G(r, r+2) \,/\, h \subset \mathbb{P}^r(\sigma) \subset H\} \cong \mathbb{P}^1$, and $[\pi] \in \Omega_{h,H}$. In this way ϕ defines Ω_π as a cone. ■

Remark 2.8 Segre variety $\pi^* \times H_\pi$ can be geometrically thought in the next way: a line $l \subset \mathbb{P}^{r+2}$ disjoint of π parametrizes the elements of H_π, so the subvariety of the grassmannian $\Omega_\pi \cap \{\sigma / \mathbb{P}^r(\sigma) \cap l \neq \emptyset\}$ is a Segre variety of the form $\pi^* \times H_\pi$. Moreover, it is the intersection of two Schubert cycles, one of them not containing $[\pi]$, so it is a hyperplane section of Ω_π.

Proposition 2.9 *Let $\Sigma \subset G(r, r+2)$ be a congruence and $\pi \subset \mathbb{P}^{r+2}$ be an r–dimensional projective subspace such that $\Sigma \subset \Omega_\pi$. Σ has order 1 if and only if $\phi^{-1}(\pi^* \times \{H\}) \cap \Sigma$ is a line for all $H \in H_\pi$. Thus the map*

(7) $$\alpha : \Sigma \setminus \{[\pi]\} \longrightarrow H_\pi \cong \mathbb{P}^1$$

given by $\alpha(\sigma) = \mathbb{P}^r(\sigma) + \pi$ defines Σ as a rational ruled surface.

Proof: Taking $H \in H_\pi$, $\alpha^{-1}(H) = \{\sigma \in \Sigma \,/\, \mathbb{P}^r(\sigma) \subset H\}$ is 1–dimensional and each one of its irreducible components is a developable family of $\mathbb{P}^r$'s (because they are contained in H), whose focal locus has dimension $\leq r$. If $\mathrm{ord}(\Sigma) = 1$, then $\dim F(\Sigma) = r$, and so the focal locus of those families must be $(r-1)$–dimensional. Equivalently, each one is a family of $\mathbb{P}^r(\sigma)$'s containing an $(r-1)$–dimensional space in H; since $\mathrm{ord}(\Sigma) = 1$, each $H \supset \pi$ can only contain one of such families (otherwise, every point in H would be focal). So $\alpha^{-1}(H) \cong \mathbb{P}^1$. Conversely, a congruence constructed in such way has clearly order 1. ■

Proposition 2.10 *Let $\Sigma \subset \Omega_\pi \subset G(r, r+2)$ be a congruence of order 1. If Σ has no fixed points, then* $\mathrm{cl}(\Sigma) \geq r$.

Proof: Consider

$$\beta : \Sigma \setminus \{[\pi]\} \xrightarrow{\phi} \pi^* \times H_\pi \xrightarrow{p} \pi^* \tag{8}$$

given by $\beta(\sigma) = \mathbb{P}^r(\sigma) \cap \pi$. In each hyperplane $H \supset \pi$, Σ consists of a pencil of $\mathbb{P}^r$'s contained in H (with base a certain $(r-1)$–dimensional projective space). If $P \in \mathbb{P}^{r+2}$ is a fixed point of Σ, necessarily $P \in \pi$, and so $\beta(\Sigma) \subset \pi^*(P) = \{h \in \pi^* \,/ P \in h\} \subset \pi^*$ will be degenerate. Thus Σ has fixed points iff $\beta(\Sigma) \subset \pi^*$ is degenerate. Now, we have two cases:

1. If $\phi(\Sigma)$ is a curve: $\mathrm{cl}(\Sigma) + 1 = \deg(\Sigma) = \deg(\phi(\Sigma)) = \#(\phi(\Sigma) \cap (h \times H_\pi + \pi^* \times H))$ where $h \in \pi^*$ and $H \in H_\pi$. Hence $\mathrm{cl}(\Sigma) + 1 = \#(\phi(\Sigma) \cap (h \times H_\pi)) + \#(\phi(\Sigma) \cap (\pi^* \times H)) \geq \deg(\beta(\Sigma)) + 1 \geq \dim(\beta(\Sigma)) + 1 = r + 1$.

2. If $\phi(\Sigma)$ is a surface: $\mathrm{cl}(\Sigma) + 1 = \deg(\Sigma) \geq \deg(\phi(\Sigma))$. The map $p : \phi(\Sigma) \longrightarrow \beta(\Sigma)$ consists of projecting from a certain $\pi^* \times H$, that contains exactly one generator of $\phi(\Sigma)$. Hence $\mathrm{cl}(\Sigma) + 1 \geq \deg(\beta(\Sigma)) + 2 \geq \dim\langle\beta(\Sigma)\rangle - 1 + 2 = r + 1$. ■

3 Case I: $F(\Sigma)$ is r–dimensional irreducible of degree 3

First examine the case $r = 1$: by (2.6) $F(\Sigma)$ is a nondegenerate curve of degree 3 in $\mathbb{P}^3$, that is a rational normal cubic, and $\Sigma = \Sigma(F(\Sigma))$. Conversely, the congruence of secant lines to a rational normal cubic has clearly order 1. Since a generic plane $\pi \subset \mathbb{P}^3$ cuts $F(\Sigma)$ in three noncollinear points, $\mathrm{cl}(\Sigma) = 3$, thus $\deg(\Sigma) = 4$. Moreover, the secant lines through a fixed point $P \in F(\Sigma)$ form a quadric cone, so Σ is not a ruled surface (the lines in $G(1,3)$ parametrize linear pencils of lines through a point). Such a quadric cone is parametrized by an irreducible conic $\Sigma(P) \subset \Omega_P$ where Ω_P is the family of lines through P. Given two points $P, Q \in F(\Sigma)$, $\Omega_P \cap \Omega_Q = [PQ]$ so $\langle \Sigma(P), \Sigma(Q) \rangle = \Omega_P + \Omega_Q = T_{G(1,3),[PQ]}$ that is 4–dimensional and $T_{G(1,3),[PQ]} \cap \Sigma = \Sigma(P) \cup \Sigma(Q)$. Hence Σ is nondegenerate in $\mathbb{P}^5 = \langle G(1,3) \rangle$. By Del Pezzo's Theorem (see for instance [5] page 525), Σ must be the Veronese surface.

By (1.6), if $r > 1$, a generic 3–dimensional subspace $L \subset \mathbb{P}^{r+2}$ provide us a projection $\rho_L : \Sigma \longrightarrow \Sigma_L \subset \mathbb{P}^5$, where Σ_L is a Veronese surface by the above. Since Σ_L is normal and ρ_L is regular and birrational, Σ will be a Veronese surface too.

$F(\Sigma)$ is an r–dimensional irreducible variety of degree 3 in $\mathbb{P}^{r+2}$. A description of such varieties can be found in [4]: Such a variety is, in general, a cone with vertex a subspace C over a variety of the same degree and codimension in a projective subspace complementary to C. If it is smooth, then it is a rational normal scroll. Thus it contains a 1–dimensional family of disjoint $\mathbb{P}^{r-1}$'s, which is only possible if $r \leq 3$.

Therefore $F(\Sigma)$ is one of the next varieties:

- the rational normal cubic in $\mathbb{P}^3$ $(r = 1)$,

- the rational normal cubic surface in $\mathbb{P}^4$, projective realization of the Blowing-Up of the plane in a point ($r = 2$),
- the rational normal cubic 3–fold in $\mathbb{P}^5$, projection of the Blowing-Up of $\mathbb{P}^3$ in a line from one of its points ($r = 3$),
- a cone with vertex a subspace C over one of the varieties described above.

Such varieties are intersection of 3 quadrics, and they contain exactly a 2–dimensional family of $(r-1)$–dimensional quadrics. These quadrics are smooth iff $F(\Sigma)$ is smooth, othercase they are cones with vertex C. Σ must therefore be the congruence of $\mathbb{P}^r$'s containing such quadrics. Conversely, a congruence constructed in this way has clearly order 1, because cutting it with a 3–dimensional space we obtain the family of secant lines to a irreducible nondegenerate curve of degree 3. Summarizing, we have:

Theorem 3.1 *Every irreducible subvariety of* $\mathbb{P}^{r+2}$ *of degree* 3 *and codimension* 2 *contains a* 2*–dimensional family of* $(r-1)$*–dimensional quadrics, and the family of* $\mathbb{P}^r$*'s containing those quadrics is a congruence of order* 1 *and class* 3*, parametrized by a Veronese surface* Σ*.* Σ *lie on the Schubert cycle* $\Omega_V = \{\sigma \in G(r, r+2) \,/\, \mathbb{P}^r(\sigma) \supset V\}$*, being* V *the singular locus of all the quadrics, and* $r - 2 \geq \dim V \geq r - 4$*. Conversely, every congruence of order* 1 *with nonlinear irreducible focal locus is constructed in this way.* ∎

Remark 3.2 Thus there are only three cases of congruences in the case II without fixed points, and the rest are cones over one of those; that is, using the isomorphism

$$\Omega_C \cong G(r - \dim C - 1, r - \dim C + 1)$$

there are only three different embeddings of the Veronese surface in a grassmannian as a congruence of bidegree $(1, 3)$: in $G(1, 3)$, in $G(2, 4)$ and in $G(3, 5)$.

4 Case II: $F(\Sigma)$ is reducible

In this case, for each $\sigma \in \Sigma$, the focal quadric in $\mathbb{P}^r(\sigma)$ is reducible, $Q(\sigma) = \mathbb{P}_1^{r-1}(\sigma) \cup \mathbb{P}_2^{r-1}(\sigma)$, and the components of $F(\Sigma)$ are the projective realizations, X and π, of the $\mathbb{P}_1^{r-1}(\sigma)$'s and of the $\mathbb{P}_2^{r-1}(\sigma)$'s, respectively. By (2.5), $\dim X = \dim \pi = r$ and π is linear.

Proposition 4.1 *Under the above assumptions,* X *is an* r*–dimensional scroll (resp. a curve, for* $r = 1$*) such that each one of its generators (resp. points, for* $r = 1$*) is contained in exactly one linear pencil of* $\mathbb{P}^r$*'s of the congruence.*

Proof: For $r = 1$, we have already seen that $\Sigma = \Sigma(X, \pi)$ in the proof of (2.5). Being $P \in X$ generic, the lines of the congruence containing P lie on the plane $\pi + P$, hence they are exactly one linear pencil.

If $r>1$, the family of $\mathbb{P}_1^{r-1}(\sigma)$'s contained in X cannot be 2–dimensional; otherwise X would be a projective space and, as Σ is nondegenerate, $\dim X\cap\pi=(r-2)$, so $X\cap\pi=\mathbb{P}_1^{r-1}(\tau)\cap\mathbb{P}_2^{r-1}(\tau)$ for all $\tau\in\Sigma$; but the family of $\mathbb{P}^{r-1}$'s in X containing $X\cap\pi$ is not 2–dimensional, a contradiction. Therefore X is a scroll and each one of its generators $\mathbb{P}_1^{r-1}(\sigma)$ is contained in an infinity of elements of Σ, all of them lying in $\mathbb{P}_1^{r-1}(\sigma)+\pi$; by (2.6) and (2.9) such elements are exactly the linear pencil $\Omega_{\mathbb{P}_1^{r-1}(\sigma),\mathbb{P}_1^{r-1}(\sigma)+\pi}=\alpha^{-1}(\alpha(\sigma))$. ■

Example 4.2 *The case* $r=1$*:* let $\pi\subset\mathbb{P}^3$ be a line, $X\subset\mathbb{P}^3$ an irreducible curve of degree n and suppose $\Sigma=\Sigma(\pi,X)$ has order 1. By (2.6), the generic plane $H\supset\pi$ contains exactly one point of X out of π (the base point of the pencil of lines of Σ contained in H). Thus π cuts X in $n-1$ points counted with multiplicity, and the projection

$$\gamma : X\longrightarrow\{H\in\mathbb{P}^{3^*}\,/\,H\supset\pi\}\cong\mathbb{P}^1$$

given by $\gamma(x)=x+\pi$ shows that X is rational and smooth out of π. Moreover, if $H\not\supset\pi$ is generic, it cuts X in n distinct points, no two collinear with $X\cap\pi$, so H contains exactly n lines of Σ; therefore $\mathrm{cl}(\Sigma)=\deg(X)=n$.

X is a projection of a rational normal curve $\Gamma_n\subset\mathbb{P}^n$ from a space $V\subset\langle x_1,\dots,x_{n-1}\rangle$, $x_1,\dots,x_{n-1}\in\Gamma_n$, disjoint of Γ_n. There are two possibilities: either $X\subset\mathbb{P}^3$ is nondegenerate, equivalently $\dim V=n-4$, or X is a plane curve, equivalently $\dim V=n-3$. Conversely, if X is such a projection of a rational normal curve, and π is a line containing the projection of $\langle x_1,\dots,x_{n-1}\rangle$, the congruence $\Sigma(\pi,X)$ has clearly order 1. ■

Remark 4.3 The curve X is smooth out of π, but it is not smooth in general. For instance, consider the congruence $\Sigma(X,\pi)$ where X is a nodal cubic in a plane L and π is a line meeting L in the double point of X.

Example 4.4 *The case* $r>1$*:* let $\Sigma\subset G(r,r+2)$ be a congruence of order 1 in the case II. In (4.1) we have shown how Σ is constructed from its focal locus $F(\Sigma)=X\cup\pi$. According to (1.11), assume that Σ has no fixed points. By each generator $\mathbb{P}_1^{r-1}(\sigma)\subset X$ passes the linear pencil $\Omega_{\mathbb{P}_1^{r-1}(\sigma),\mathbb{P}_1^{r-1}(\sigma)+\pi}$. Let us denote $\Sigma(\pi,X)=\Sigma$. By (2.6), the projection $\gamma:X\longrightarrow\{H\in\mathbb{P}^{r+2^*}\,/\,H\supset\pi\}\cong\mathbb{P}^1$ given by $\gamma(x)=x+\pi$ forces X to be a rational scroll and smooth out of π. In order to compute $\mathrm{cl}(\Sigma)$, take a generic 3–dimensional subspace $L\subset\mathbb{P}^{r+2}$ and consider the section Σ_L, whose focal locus is $F(\Sigma_L)=(X\cap L)\cup(\pi\cap L)$. By (1.6) and the above example, we get: $\mathrm{cl}(\Sigma)=\mathrm{cl}(\Sigma_L)=\deg(X\cap L)=\deg(X)$.

Being $\deg X=n$, X is a projection of a rational normal scroll $R\subset\mathbb{P}^{n+(r-1)}$ of the same degree. The center V of such projection (p_V) is disjoint of R, and is contained in the space generated by $p_V^*(\pi\cap X)$. Moreover $\dim V=(n+(r-1)-\dim\langle X\rangle-1)$, being $\dim\langle X\rangle=r+2$ or $r+1$. Conversely, every r–dimensional rational normal scroll can be projected to $\mathbb{P}^{r+2}$ in this way, obtaining the focal scroll of a congruence of order 1 in the case II without fixed points. ■

The next proposition provides a criterion for the smoothness of a congruence in the case II.

Theorem 4.5 *Let $\Sigma = \Sigma(\pi, X) \subset G(r, r+2)$ be a congruence in the case II. Σ is a rational ruled surface of degree $\deg(\Sigma) = \deg(X) + 1$ contained in $\Omega_\pi = \{\sigma \in G(r, r+2) \,/\, \dim(\mathbb{P}^r(\sigma) \cap \pi) \geq r-1\}$. If Σ has no fixed points, by $[\pi]$ are passing at most $n-r$ generators of Σ (exactly $n-1$ if $r=1$) counted with multiplicity. The singular locus of Σ consists of, at most, $[\pi]$ and the multiple generators of Σ passing by it. Hence Σ is smooth if and only if $\pi \cap X$ contains at most one generator of X. In particular, if $n \leq r+1$, Σ is always smooth, and if $r=1$, then Σ is smooth iff $n \leq 2$.*

Proof: If $r=1$, consider $\gamma : X \longrightarrow \{H \in \mathbb{P}^{3*} \,/\, H \supset \pi\} \cong \mathbb{P}^1$ defined in (4.2). It is an isomorphism in $X \setminus \{[\pi]\}$ and puts in correspondence the $n-1$ points (counted with multiplicity) of $\pi \cap X$ with the planes by π containing pencils of lines of Σ with base point in $\pi \cap X$; equivalently $[\pi]$ belongs to those pencils. Thus $[\pi]$ will be contained in $n-1$ generators of Σ counted with multiplicity.

If $r > 1$ and Σ has no fixed points, consider the projection $p_V : R \longrightarrow X$ given in (4.4), whose center V is contained in $\langle p_V^*(\pi \cap X)\rangle$. Clearly $p_V^*(\pi \cap X) \sim H - F$ where $\sim$ denotes linear equivalence, H the hyperplane section of R and F one of its generators. The generators of R contained in $p_V^*(\pi \cap X)$ are in correspondence with the generators of Σ passing by $[\pi]$. Thus, it is enough to show that $p_V^*(\pi \cap X)$ contains at most $n-r$ generators counted with multiplicity: a hyperplane section $H \subset R$ is a divisor of the form $H = C + F_1 + \ldots + F_k$, where F_i are generators and $C = \overline{\bigcup_{F \neq F_1, \ldots, F_k} (F \cap H)}$ is an irreducible $(r-1)$–dimensional scroll with disjoint generators (since R is normal); then $\dim\langle C\rangle \geq 2(r-2)+1$ and $\deg(C) \geq \operatorname{codim}(C \subset \langle C\rangle) + 1 \geq r-1$. Since $\deg(H) = n$, $k \leq n-(r-1) = n-r+1$, which is our claim. ■

Being $n_1, \ldots, n_r \geq 1$ integers, $n = n_1 + \ldots + n_r$, $r > 1$, let $R(n_1, \ldots, n_r) \subset \mathbb{P}^{n+(r-1)}$ denote the rational normal scroll generated by the rational normal curves $\Gamma_{n_1}, \ldots, \Gamma_{n_r}$ in disjoint spaces (given r isomorphisms $\nu_i : \mathbb{P}^1 \longrightarrow \Gamma_{n_i}$, $i = 1, \ldots, r$, $R(n_1, \ldots, n_r)$ is the scroll generated by the spaces $\langle \nu_1(t), \ldots, \nu_r(t)\rangle$, with $t \in \mathbb{P}^1$); H denotes its hyperplane section and F one of its generators.

Proposition 4.6 *The rational normal scroll $R(n_1, \ldots, n_r)$ can be projected to $\mathbb{P}^{r+2}$ providing the focal scroll X of a nonsingular congruence $\Sigma(\pi, X)$. Hence, given $n \geq r$, there exist smooth congruences $\Sigma \subset G(r, r+2)$ of order 1 and class n without fixed points. Moreover, if $n \geq r+2$ and if $\max(n_1, \ldots, n_r) \geq 3$, $R(n_1, \ldots, n_r)$ can be projected to $\mathbb{P}^{r+2}$ providing the focal scroll of a singular congruence $\Sigma(\pi, X)$. Hence, if $n \geq r+2$, there exist singular congruences $\Sigma \subset G(r, r+2)$ of order 1 and order n.*

Proof: As we have seen in the proof of (4.5), the singularity of $\Sigma(\pi, X)$ depends of the number of generators of $R := R(n_1, \ldots, n_r)$ contained in $p_V^*(\pi \cap X) \sim H - F$; we will thus have to choose properly a divisor $C \sim H - F$ in each case.

For the first part, it is sufficient to show that $|H - F|$ contains irreducible divisors: the trace of $|H|$ over F is the complete linear series $|\mathcal{O}_F(H \cap F)|$, so we have an exact sequence

$$0 \to H^0(\mathcal{O}_R(H-F)) \longrightarrow H^0(\mathcal{O}_R(H)) \longrightarrow H^0(\mathcal{O}_F(1)) \to 0$$

and $h^0(\mathcal{O}_R(H-F)) = n$; moreover the trace of $|H-F|$ over another generator F' is $|\mathcal{O}_{F'}(H\cap F')|$, so the exact sequence

$$0 \to H^0(\mathcal{O}_R(H-F-F')) \longrightarrow H^0(\mathcal{O}_R(H-F)) \longrightarrow H^0(\mathcal{O}_{F'}(1)) \to 0$$

provides $h^0(\mathcal{O}_R(H-F-F')) = n-r$. Making F' vary in R, the set of reducible elements of $|H-F|$ has dimension $n-r$, hence it is a proper closed subset.

For the second part, we will show that $k = \max(n_1, \dots, n_r)$ is the greatest number of generators of R contained in a hyperplane (thus, if $k \geq 3$, a divisor in $|H-F|$ can contain $k-1 \geq 2$ generators of R): let $F_1, \dots, F_l$ be generators of R, $F_i = \langle P_i^1, \dots, P_i^r\rangle$, $P_i^j \in \Gamma_{n_j}$, $i = 1, \dots, l$. They are contained in a hyperplane iff $F_1+\ldots+F_l \neq \mathbb{P}^{n+r-1}$, iff $\langle P_1^1, \dots, P_l^1\rangle+\ldots+\langle P_1^r, \dots, P_l^r\rangle \neq \mathbb{P}^{n+r-1}$, iff $l \leq n_j$ for some $j = 1, \dots, r$, iff $l \leq \max(n_1, \dots, n_r)$, which is our claim. ■

The next proposition deals with the existence of congruences with assigned linearly normal model. We will use the notation of [6] chapter V §2.

Proposition 4.7 *Given $n \geq 1$ and $1 \leq r \leq n$, there exist congruences without fixed points $\Sigma \subset G(r, r+2)$ in the case II of order 1 and class n, with given invariant e, $0 \leq e \leq n-1$, $n-e \equiv 1 (\mathrm{mod}\, 2)$.*

Proof: Let X_e be the geometrically ruled surface of invariant e, C_0 its minimal directrix ($C_0^2 = -e$) and F one of its generators. Suppose $0 \leq e \leq n-1$ and $n-e \equiv 1 (\mathrm{mod}\, 2)$ and consider the morphisms

$$\psi_1 : X_e \longrightarrow \mathbb{P}^1, \qquad \psi_2 : X_e \longrightarrow \mathbb{P}^n$$

given by the linear systems $|F|$ and $|C_0 + (n+e-1)/2\, F|$, respectively. Since $(n-e-1)/2 \geq e$ the second one is base point free, so both maps are regular. Regard the composition

$$\psi : X_e \stackrel{\psi_1\times\psi_2}{\longrightarrow} \mathbb{P}^1 \times \mathbb{P}^n \stackrel{i}{\longrightarrow} \Omega_\pi \subset \mathbb{P}^{2n+2}$$

where $\pi \subset \mathbb{P}^{n+2}$ is an n–dimensional subspace, and we use the identification of Ω_π with the cone with vertex $[\pi]$ over the Segre variety given in (2.7). If $H \subset \mathbb{P}^{2n+2}$ is a hyperplane, then $\psi^{-1}(H) = C_0 + (n+e+1)/2\, F$ and $\psi^*(|H|) = |C_0 + (n+e+1)/2\, F|$. Hence $\psi(X_e)$ is a rational normal ruled surface of degree $n+1$ and invariant e. Since $\psi_1^{-1}(x) \sim F$, the condition of (2.9) holds, so Σ has order 1. Each generator $F \subset \psi(X_e)$ parametrizes a pencil of $\mathbb{P}^n$'s not containing π, equivalently, their focal locus does not lie on π, so $\psi(X_e)$ is in the case II, and it is smooth since $[\pi] \notin \psi(X_e)$.

Moreover, for $r < n$ it is sufficient to take the generic projections of $\psi(X_e)$ according to (1.6).■

Remark 4.8 Moreover, every congruence in the case II is expected to be a projection of one of those constructed above, because, as rational surfaces, those are their linearly normal models.

5 Case III: $F(\Sigma)$ is an r-dimensional projective space

$F(\Sigma)$ is the only r-dimensional projective subspace $\pi \subset \mathbb{P}^{r+2}$ such that $\Sigma \subset \Omega_\pi = \{\sigma \in G(r, r+2) \,/\, \dim \mathbb{P}^r(\sigma)\cap\pi \geq r-1\}$. Moreover, in this case, $2(\mathbb{P}^{r-1}(\sigma)) := 2(\mathbb{P}^r(\sigma)\cap\pi)$ is the focal quadric

of Σ at $\mathbb{P}^r(\sigma)$. The fibers of the map

$$\alpha : \Sigma \setminus \{[\pi]\} \longrightarrow H_\pi \cong \mathbb{P}^1$$

defined in (2.9) parametrize pencils whose base loci lie in $F(\Sigma) = \pi$, so they all contain $[\pi]$. Therefore Σ *is a rational cone with vertex* $[\pi]$. Consider the projection $\beta : \Sigma \setminus \{[\pi]\} \longrightarrow \pi^*$ defined in (2.10), $\beta(\sigma) = \mathbb{P}^r(\sigma) \cap \pi$: since the fibers of this map are $\beta^{-1}(\mathbb{P}^{r-1}(\sigma)) = \alpha^{-1}(\mathbb{P}^r(\sigma) + H)$, $\beta(\Sigma)$ is a curve. Thus we have a birrational map

$$\begin{array}{ccc} H_\pi & \xrightarrow{\quad\Lambda\quad} & \beta(\Sigma) \subset \pi^* \\ \\ H & \longmapsto & \bigcap_{\sigma\in\alpha^{-1}(H)} \mathbb{P}^r(\sigma) \end{array} \tag{9}$$

such that $\Lambda \circ \alpha = \beta$ and so Σ is characterized by Λ in the next way:

$$\Sigma = \bigcup_{H\supset\pi} \Omega_{\Lambda(H),H} \tag{10}$$

being $\Omega_{\Lambda(H),H} = \{\sigma \in G(r, r+2) \,/\, \Lambda(H) \subset \mathbb{P}^r(\sigma) \subset H\}$.

$\beta(\Sigma)$ parametrize a family of hyperplanes in π, hence it is developable, and we can say how it is constructed: suppose Σ has no fixed points (otherwise see (1.11)); $\beta(\sigma)$ is the family of $(r-1)$–osculating hyperplanes to a nondegenerate curve $C \in \pi$, which is birrational to $\beta(\Sigma)$. Thus we have another birrational map:

$$\begin{array}{ccl} H_\pi & \xrightarrow{\quad\Lambda'\quad} & C \subset \pi \\ \\ H & \longmapsto & \Lambda'(H) = (r-1)\text{–th focal locus of } \beta(\Sigma) \text{ at } \Lambda(H) \end{array}$$

that characterizes:

$$\Sigma = \bigcup_{H\supset\pi} \Omega_{T_{r-1,C,\Lambda'(H)},H} \tag{11}$$

where $T_{r-1,C,\Lambda'(H)}$ denotes the $(r-1)$-osculating hyperplane to C at $\Lambda'(H)$.

Let ϕ $(= \alpha \times \beta)$ be the map defined in (2.7); Σ is a cone with vertex $[\pi]$ over the curve $\phi(\Sigma \setminus \{[\pi]\}) \subset \pi^* \times H_\pi$, which is a hyperplane section of Σ. Since this curve is the graph of the map $\Lambda : H_\pi \longrightarrow \pi^*$, $\phi(\Sigma \setminus \{[\pi]\})$ is an smooth curve of degree $n+1$ (where $n := \deg(\beta(\Sigma)) = \deg(\Lambda(H_\pi)))$. Thus $\deg(\Sigma) = \deg(\phi(\Sigma \setminus \{[\pi]\})) = n+1 = \deg(\Lambda(H_\pi)) + 1$ and we conclude $\mathrm{cl}(\Sigma) = n$. Moreover the singular locus of Σ is only the vertex $[\pi]$. Summarizing, we have:

Theorem 5.1 *For every congruence $\Sigma \subset G(r, r+2)$ in the case III without fixed points there exist a regular map*

$$\Lambda : H_\pi \longrightarrow \pi^*$$

(being $\pi = F(\Sigma)$ and H_π the pencil of hyperplanes containing π) with nondegenerate image $\Lambda(H_\pi)$ such that:

$$\Sigma = \bigcup_{H\supset\pi} \Omega_{\Lambda(H),H}$$

being $\Omega_{\Lambda(H),H} = \{\sigma \in G(r, r+2) \,/\, \Lambda(H) \subset \mathbb{P}^r(\sigma) \subset H\}$. Moreover, Σ is a cone of degree $\deg(\Lambda(H_\pi)) + 1$ $(\mathrm{cl}(\Sigma) = n)$, with vertex $[\pi]$ over an smooth curve. Conversely, a congruence constructed in this way is in the case III. ■

The curve $\beta(\Sigma) \subset \pi^*$ is the projection of its normal model $\Gamma_n \subset (\mathbb{P}^n)^*$ from a subspace $V \subset (\mathbb{P}^n)^*$, hence the map Λ factorizes through Γ_n. Regard $\pi \subset \mathbb{P}^n$, and everything contained in a projective space $\mathbb{P}^{n+2}$ such that π is obtained cutting $\mathbb{P}^n$ with the $(r+2)$–dimensional space containing the congruence Σ (allowing us to identify the hyperplanes in $\mathbb{P}^{r+2}$ containing π with the hyperplanes in $\mathbb{P}^{n+2}$ containing $\mathbb{P}^n$), we will have a map $\Lambda_n : H_{P^n} \longrightarrow (\mathbb{P}^n)^*$ providing a congruence $\Sigma_n \subset G(n, \mathbb{P}^{n+2})$ whose projection to $G(r, \mathbb{P}^{r+2})$ is Σ. Hence we have a commutative diagram

$$\begin{array}{ccccc}
\Gamma_n \subset (\mathbb{P}^n)^* & \longleftarrow & (\mathbb{P}^n)^* \times H_{P^n} & \xleftarrow{\phi_n} & \Omega_{P^n} \supset \Sigma_n \\
\downarrow \quad \downarrow & & \downarrow & & \downarrow \quad \downarrow \rho_{\mathbb{P}^{r+2}} \\
\beta(\Sigma) \subset \pi^* & \longleftarrow & (\pi)^* \times H_\pi & \xleftarrow{\phi} & \Omega_\pi \supset \Sigma
\end{array}$$

We have thus proved:

Theorem 5.2 *Being $n \geq 1$, there exist a congruence $\Sigma_n \subset G(n, n+2)$ constructed in the next way: given $L \subset \mathbb{P}^{n+2}$ an n–dimensional projective space and $\Lambda : \mathbb{P}^1 \cong H_L \longrightarrow L^*$ a n–th Veronese embedding, $\Sigma_n := \bigcup_{H \supset L} \Omega_{\Lambda(H),H}$. Σ_n is a rational normal cone of degree $n+1$. Every congruence $\Sigma \subset G(r, r+2)$ of order 1 and class n without fixed points in the case III can be obtained cutting Σ_n with a suitable projective subspace $\mathbb{P}^{r+2}$, that is:*

$$\Sigma_n \subset G(n, n+2) \overset{\rho_{\mathbb{P}^{r+2}}}{\longrightarrow} \Sigma \subset G(r, r+2).$$

■

6 Smoothness of a congruence of order 1

According to (1.11), in order to study the smoothness of congruences of order 1 we only need to consider congruences without fixed points, translating the results to the general case. Summarizing (3.1), (4.5), (4.6) and (5.1), we have the next theorems:

Theorem 6.1 *Let r, s be two integers such that $r \geq 1$ and $-1 \leq s \leq r-1$. If $\Sigma \subset G(r, r+2)$ is a surface of order 1 and class n such that $\dim(T(\Sigma)) = s$, then $n \geq r - s - 1$. Furthermore:*

- *If $s = r-1$, Σ is a plane.*
- *For every $s \leq r-2$, there exist singular surfaces $\Sigma \subset G(r, r+2)$ of bidegree $(1, n)$ such that $\dim(T(\Sigma)) = s$.*

If $s = r-2$, then:

1. *If $n = 1, 2$ and Σ is not a cone, Σ is smooth.*
2. *If $n \geq 4$, Σ is singular.*

3. *If $n = 3$, there exist smooth and singular surfaces $\Sigma \subset G(r, r+2)$ of bidegree $(1, 3)$ such that* $\dim T(\Sigma) = s$.

If $-1 \leq s \leq r-3$, then:

- *If $n = r - s - 1$ or $n = r - s$ and Σ is not a cone, Σ is smooth.*
- *If $n \geq r - s + 1$, there exist smooth and singular surfaces $\Sigma \subset G(r, r+2)$ of bidegree $(1, n)$ such that* $\dim T(\Sigma) = s$.

Proof: Let $\Sigma \subset G(r, r+2)$ be a congruence of order 1 and class n such that the dimension of its fixed locus is $\dim(T(\Sigma)) = s$; if $s = r-1$, Σ is the family of $\mathbb{P}^r$'s containing $T(\Sigma)$, which is parametrized by a plane; in this case $n = 0 = r - s - 1$; suppose now that $s \leq r-2$. Applying (1.11), Σ can be projected isomorphically to a congruence $\Sigma' \subset G(r-s-1, r-s+1)$ without fixed points, having order 1 and class n; if Σ' is in the case I, then $n = 3$ and $r - s - 1 \in \{1, 2, 3\}$, and then $n \geq r - s - 1$; otherwise (2.10) shows that $n = \mathrm{cl}(\Sigma') \geq r - s - 1$. Moreover for every $n \geq r-s-1$, there exist a congruence $\Sigma' \subset G(r-s-1, r-s+1)$ of class n in the case III without fixed points (see (5.2)), which is parametrized by a cone. Excepting those congruences, the rest of the theorem is consequence of (4.5) and (3.1) and (4.6). ■

Theorem 6.2 *Being $r \geq 1$, the only smooth surfaces in $G(r, r+2)$ of bidegree $(1, n)$ are:*

1. *The plane parametrizing the family of $\mathbb{P}^r$'s containing a fixed $\mathbb{P}^{r-1}$ ($n = 0$).*
2. *The Veronese surface, described in (3.1), for $r = 1, 2, 3$ ($n = 3$); embedding $G(1, 3)$, $G(2, 4)$ or $G(3, 5)$ through the map given in (1.11), it can be consider in every $G(r, r+2)$.*
3. *The rational ruled surfaces $\Sigma(\pi, X)$ described in Section 4 ($n = \deg X$), and verifying that π contains at most one generator of X; this condition is always verified for $n = r, r+1$.*
4. *The surfaces described in 2. and 3. embedded in a higher dimension grassmannian $G(r, r+2)$ through the isomorphism exposed in (1.11).* ■

Let us finally enumerate the smooth congruences of order 1 in $G(1, 3)$, recovering a Ziv Ran's result given in [10]. We remark that, for a congruence $\Sigma(\pi, X)$ in $G(1, 3)$, π is a line cutting the curve X in $n-1$ points; hence, imposing the smoothness condition of (4.5), we have $n - 1 \leq 1$, that is $\deg X \leq 2$.

Corollary 6.3 *The only smooth congruences in $G(1, 3)$ of order 1 are:*

1. *The plane parametrizing the lines passing by a fixed point (bidegree (1,0)).*
2. *The Veronese surface $\Sigma(C)$ parametrizing the secant lines to a rational normal cubic C (bidegree (1,3)).*
3. *The quadric $\Sigma(\pi, \pi')$ parametrizing the lines cutting two given lines π and π' (bidegree (1,1)).*
4. *The rational normal cubic $\Sigma(\pi, C)$ parametrizing the lines cutting a given line π and a given irreducible conic C meeting in a point $P = \pi \cap C$ (bidegree (1,2)).* ■

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